ALGEBRAS, LIE ALGEBRAS AND SO(2)

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Reference: Robert D. Klauber, *Student Friendly Quantum Field Theory, Vol. 2*, Section 2.3.

An algebra is defined as a set with two binary operations and a scalar operation. For a set of matrices, we could have the binary operations of matrix addition, matrix multiplication and the operation of multiplication by a scalar. Alternatively, we could have matrix addition, matrix commutation and scalar multiplication. The latter choice is known as a *Lie algebra*, and is central to quantum field theory.

An algebra must satisfy closure under both binary operations and also under multiplication by a scalar. Further, the second binary operation must be distributive with respect to the first. In the case of a Lie algebra, this means that the two binary operations are $A+B$ and $[A,B]$, where the brackets indicated a commutator. The commutator $[A,B]$ must give another matrix that is in the set. The distributive property means that

$$[A,B+C]=[A,B]+[A,C] \tag{1}$$

As an example of a Lie algebra, we return to the specific example of 2-d rotation. We treated the general case of rotation in n dimensions earlier, but a simpler approach might help us to understand the idea behind a Lie algebra.

2-d rotation can be represented as a Lie group, in that such rotations satisfy the group axioms and depend on a continuous parameter (the rotation angle). With the binary operation of addition of rotation angles, the set of rotations is closed (the sum of any two rotations gives another rotation), has an identity (a rotation angle of zero), an inverse (rotation by $-\theta$ is the inverse of rotation by θ), and is associative. We can represent the group by a matrix

$$\hat{M}(\theta)=\left[\begin{array}{cc}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{array}\right] \tag{2}$$

A rotation is performed by multiplying a 2-d vector by $\hat{M}(\theta)$.

The matrix can be expanded in a Taylor series as

$$\hat{M}(\theta)=\hat{M}(0)+\theta\hat{M}'(0)+\frac{\theta^{2}}{2!}\hat{M}''(0)+\ldots \tag{3}$$

For small angles, we can approximate this by

$$\hat{M}(\theta) \approx \hat{M}(0) + \theta \hat{M}'(0) \tag{4}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

This can be written as

$$\hat{M}(\theta) \approx I + i\theta \hat{X} \tag{6}$$

where

$$\hat{X} \equiv -i\hat{M}'(0) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{7}$$

[The factor of $i = \sqrt{-1}$ is inserted by convention.]

The matrix $\hat{X}$ is an algebra (actually, a Lie algebra) as we can see by checking it satisfies the axioms. We consider the set of matrices that are scalar multiples of $\hat{X}$ so that, for real θ_k:

$$\hat{X}_k = \theta_k \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{8}$$

Then the sum of two such matrices is

$$\hat{X}_1 + \hat{X}_2 = \theta_1 \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} + \theta_2 \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{9}$$

$$= (\theta_1 + \theta_2) \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{10}$$

which is of the same form as 8.

With commutation as the second binary operation, we have

$$\left[\hat{X}_1, \hat{X}_2\right] = \theta_1\theta_2 \left[\hat{X}, \hat{X}\right] = \theta_1\theta_2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

where this follows because any matrix commutes with itself. The commutator is thus equal to the zero matrix, which is of the form 8 with $\theta_k = 0$.

$\hat{X}_k$ also satisfies the distributive property, since

$$\left[\hat{X}_1, \hat{X}_2 + \hat{X}_3\right] = \hat{X}_1\hat{X}_2 + \hat{X}_1\hat{X}_3 - \hat{X}_2\hat{X}_1 - \hat{X}_3\hat{X}_1 \tag{12}$$

$$= \left[\hat{X}_1, \hat{X}_2\right] + \left[\hat{X}_1, \hat{X}_3\right] \tag{13}$$

[Actually, all such commutators equal the zero matrix because of 11.]

The Lie group $SO(2)$ can be *generated* from the Lie algebra by using $\hat{X}$ in 7 as a *generator*. Working out the derivatives in 3 we have

$$\begin{aligned}
\hat{M}(0) &= I \\
\hat{M}'(0) &= i\hat{X} \\
\hat{M}''(0) &= \begin{bmatrix} -\cos 0 & \sin 0 \\ -\sin 0 & -\cos 0 \end{bmatrix} = -I \\
\hat{M}'''(0) &= \begin{bmatrix} \sin 0 & \cos 0 \\ -\cos 0 & \sin 0 \end{bmatrix} = -i\hat{X}
\end{aligned} \tag{14}$$

After the last term, the derivatives repeat in a cycle of length 4, so we have $\left\{I, i\hat{X}, -I, -i\hat{X}, I, i\hat{X}, -I, -i\hat{X}\ldots\right\}$. This probably isn't terribly surprising, since we are just reproducing the Taylor series for $\sin\theta$ and $\cos\theta$. That is, we get

$$\hat{M}(\theta) = \begin{bmatrix} 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots & -\theta+\frac{\theta^3}{3!}-\frac{\theta^5}{5!}+\ldots \\ \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots & 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{15}$$

However, we can observe that

$$\begin{aligned}
\left(i\hat{X}\right)^0 &= I \\
\left(i\hat{X}\right)^1 &= i\hat{X} \\
\left(i\hat{X}\right)^2 &= -\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \\
\left(i\hat{X}\right)^3 &= i\hat{X}\left(i\hat{X}\right)^2 = -i\hat{X}
\end{aligned} \tag{16}$$

This pattern also repeats with a length of 4, so we have

$$\hat{M}(\theta) = I + \left(i\theta\hat{X}\right) + \frac{\left(i\theta\hat{X}\right)^2}{2!} + \frac{\left(i\theta\hat{X}\right)^3}{3!} + \ldots \tag{17}$$

$$= e^{i\theta\hat{X}} \tag{18}$$

That is, we can generate the Lie group for $SO(2)$ by exponentiating the generator. This turns out to be a general property relating a Lie algebra to its associated Lie group. The example shown here is called a 'trivial' Lie algebra, because there is only one generator. Higher order Lie groups have more than one generator, and in general, these generators do not commute, so things get considerably more complicated.

PINGBACKS

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